# ON A METHOD OF SOLUTION OF THE AXISYMMETRIC CONTACT PROBLEM OF THE THEORY OF ELASTICITY

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*PMM Vol.25, No.1, 1961, pp. 76-85*

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*(Received November 14, 1960)*

In a paper by Korenev [1] the problem of the indentation of a general-type elastic foundation by a circular planar stamp was reduced to a pair of integral equations for a certain auxiliary function.

In the case of the contact stress this problem is directly reduced in the present paper to a Fredholm integral equation of the first kind. Further, for certain types of elastic foundations, in particular a homogeneous elastic half-space, this integral equation is simply transformed into a Wiener-Hopf equation which admits an exact solution.

It is shown that in certain cases the method presented allows one to obtain an exact solution of the axisymmetric contact problem, with the surface structure of the contacting bodies taken into account [2].

**1.** The results which are obtained here are based on a formula which allows one to compute the settlement $w(r)$ of the surface points of an elastic foundation which is subjected to a vertical line load distributed over the circumference of a circle. It is not difficult to derive this formula for a foundation of a very general type. One needs only to integrate the relation

$$w_0(r) = \int_0^\infty f_0(t) J_0(rt)\, dt \tag{1.1}$$

Here $J_0(x)$ is a Bessel function of the first kind, $w_0(r)$ is the settlement of a surface point of the foundation located at a distance $r = \sqrt{(x^2 + y^2)}$ from the point of application of a unit load.

One is easily convinced, for example, that for a homogeneous elastic half-space

$$f_0(t) = (1 - \mu_0^2)(\pi E)^{-1}$$

For a half-space with a modulus of elasticity which varies according to a power law $E = E_\nu z^\nu$

$$f_0(t) = \frac{\Gamma(1/2 - \nu/2)}{\pi D_\nu \Gamma(1/2 + \nu/2)} \left(\frac{t}{2}\right)^\nu \left(D_\nu = \frac{\alpha_0}{E_\nu}\right) \qquad (1.2)$$

A table for the coefficient $a_0$ has been compiled by Klein [3 ]. It can be verified further that the relation (1.1) is also valid for an elastic layer ($0 \leqslant z \leqslant h$). Likewise, it is valid for a half-space with a modulus of elasticity which varies according to the law [4 ]

$$E = E_0 \exp(\gamma z)$$

If the foundation is subjected to the load $p(x, y)$ then the settlement is determined by the formula

$$w(x, y) = \int_0^\infty f_0(t)\,dt \iint_{-\infty}^{\infty} J_0\left(t\sqrt{(x-\xi)^2 + (y-\eta)^2}\right) p(\xi, \eta)\,d\xi\,d\eta \qquad (1.3)$$

A load of unit magnitude, distributed over the circumference of a circle of radius $\rho$, can be represented in the form

$$p(x, y) = \delta(r - \rho) = \delta\left(\sqrt{x^2 + y^2} - \rho\right) \qquad (1.4)$$

where $\delta(x)$ is the delta function.

We shall denote by $w(r, \rho)$ the settlement of the foundation under the load (1.4).

Substituting from (1.4) into (1.3), making the change of variables

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad \xi = \rho\cos\theta, \quad \eta = \rho\sin\theta$$

and using the well-known properties of the delta function, we obtain

$$w(r, \rho) = \rho \int_0^\infty f_0(t)\,dt \int_0^{2\pi} J_0\left(t\sqrt{r^2 - 2\rho r\cos(\varphi + \theta) + \rho^2}\right) d\theta$$

Finally, expanding the integrand in the second integral into a series of Bessel functions and carrying out the integration, we find

$$w(r, \rho) = 2\pi\rho \int_0^\infty f_0(t) J_0(rt) J_0(\rho t)\,dt \qquad (1.5)$$

For the case of a homogeneous elastic half-space this formula coincides with the formula deduced by Egorov by other means.

The formula obtained permits one to reduce the axisymmetric contact problem, both in the case of the circle and of the ring, to the following integral equation (tangential interaction along the contact surface is not taken into account):

$$\int_{a}^{a} w(r,\rho)\, p(\rho)\, d\rho = w(r) \qquad (a_0 \leqslant r \leqslant a) \tag{1.6}$$

In the case of a circular contact region ($a_0 = 0$), and for certain types of elastic foundations, a straightforward transformation takes Equation (1.6) into a Wiener-Hopf equation for which it is not difficult to find an exact solution.

However, it seems to us that in the case of an approximate solution it is also advantageous to be able to reduce the problem of finding the contact stresses directly to an equation of the form (1.6). We also note that the relation (1.6) obtained above is convenient for the calculation of the settlement under a planar circular or annular flexible foundation.

**2.** We shall show that the contact problem for a circular region in the case of homogeneous half-space or a half-space which satisfies (1.2) can be reduced to the solution of a Wiener-Hopf integral equation. In doing this we shall pose the problem in a somewhat wider sense than in the preceding section.

We consider two contacting elastic bodies with different elastic properties and close to half-spaces in shape. Then, following Shtaerman [2, p. 175], we may write

$$\alpha = w_1(r) + z_1(r) + z_2(r) - w_2(r) \tag{2.1}$$

where $a$ is the approach of the elastic bodies upon compression, $z = z_1(r)$ and $z = z_2(r)$ are the equations of the surfaces which bound the compressed bodies (the first body is defined as the body into whose interior passes the positive $z$-semiaxis), $w_1(r)$ and $w_2(r)$ are the vertical elastic displacements of the points of contact. As usual, we shall make the approximation that the displacements $w_1$ and $w_2$ are the same as if the pressure induced over the surface of contact were acting on upper and lower elastic half-spaces with the same elastic properties as those of the compressed bodies.

Then, assuming perfectly smooth bodies, we may use Formula (1.5) for the calculation of $w_1$ and $w_2$. For an elastic foundation of type (1.2) a suitable representation is of the form

$$w(r,\rho) = c\rho^{-\nu} k_\nu(r/\rho), \qquad k_\nu(z) = \int_0^\infty J_0(s) J_0(sz) s^\nu \, ds \tag{2.2}$$

where

$$c = \frac{2^{1-\nu}}{D_\nu} \frac{\Gamma(1/2 - \nu/2)}{\Gamma(1/2 + \nu/2)}$$

In the case of a homogeneous elastic half-space one should place into Formula (2.2)

$$\nu = 0, \qquad c = \frac{2(1-\mu_0^2)}{E} \tag{2.3}$$

Retaining the previous notation $p(r)$ for the contact stress, one may, in view of (2.2), write

$$w_{1,2} = \pm c_{1,2} \int_0^a k_\nu(r/\rho) \rho^{-\nu} p(\rho) \, d\rho$$

Substituting this expression for $w_1$ and $w_2$ into (2.1) we arrive at the integral equation

$$\int_0^a k_\nu(r/\rho) \rho^{-\nu} p(\rho) \, d\rho = f(r), \qquad 0 \leqslant r \leqslant a \tag{2.4}$$

where

$$f(r) = \frac{\alpha - z_1(r) - z_2(r)}{c_1 + c_2} \tag{2.5}$$

As a result of the change of variables

$$r = ae^{-x}, \qquad \rho = ae^{-\xi}, \qquad \chi(\xi) = ae^{-\xi} p(ae^{-\xi}) \tag{2.6}$$

the integral equation (2.4) passes into a Wiener-Hopf integral equation of the first kind:

$$\int_0^\infty l(x-\xi) \chi(\xi) = g(x), \qquad 0 \leqslant x < \infty \tag{2.7}$$

In the case at hand

$$l(z) = e^{-\nu z} k_\nu(e^{-z}), \qquad g(x) = e^{-\nu x} a^\nu f(ae^{-x}) \tag{2.8}$$

The solution of Equation (2.7) is easily constructed by a method which has already been applied by us [6,7] and which was presented at the All-Union Congress of Theoretical and Applied Mechanics.

In this method it is first necessary to find a solution of the equation with a special right-hand side

$$\int_0^\infty l(x-\xi)\,\chi_\zeta(\xi)\,d\xi = e^{i\zeta x}, \qquad \operatorname{Im}\zeta \geqslant 0 \tag{2.9}$$

and then, expressing $g(x)$ as a generalized Fourier integral [8, p. 11]

$$g(x) = \frac{1}{2\pi}\int_\gamma G(w)\,e^{iwx}\,dw, \qquad G(w) = \int_0^\infty g(x)\,e^{-iwx}\,dx \tag{2.10}$$

one may obtain the solution of Equation (2.7) in the form

$$\chi(x) = \frac{1}{2\pi}\int_\gamma G(\zeta)\,\chi_\zeta(x)\,d\zeta \tag{2.11}$$

However, the solution of the key equation (2.9) is given by the formula

$$\chi_\zeta(x) = \frac{i}{2\pi}\int_\gamma \frac{\psi_+(w)\,\psi_-(-\zeta)}{w+\zeta}\,e^{-iwx}\,dw \tag{2.12}$$

In this case the functions $\psi_\pm(w)$ should be regular and different from zero in the upper and lower half-planes, respectively, (excluding the point at infinity) and should satisfy the relation

$$\left[L(u) = \int_{-\infty}^{\infty} l(t)\,e^{iut}\,dt\right]^{-1} = \psi_+(u)\,\psi_-(u) \qquad (-\infty < u < \infty) \tag{2.13}$$

At infinity the functions $\psi_\pm(w)$ should behave in the following manner:

$$\psi_\pm(w) = O(w^\mu), \qquad \operatorname{Im} w \gtrless 0, \qquad \mu < 1 \tag{2.14}$$

The contour of integration is understood to be a line $(\infty, -\infty)$ parallel to the real axis in the lower half-plane and located a sufficiently small distance away from it; more exactly, such that all singular points of the integrand lie below the ingration contour.

The problem of finding the functions corresponding to the given function carries the name of the factorization problem, and has a solution for a sufficiently wide class of functions which are continuous over the entire interval $(\infty, -\infty)$ [9].

By direct substitution (compare [6,10]) it can be verified that the function (2.12) satisfies the equation (see also [9, p. 109]).

**3.** In the case at hand, the Fourier transform of the kernel of the integral equation (2.7) can be computed in finite form

$$L(u) = \frac{\Gamma(^1/_2\nu - ^1/_2 iu)}{2^{1-\nu}\Gamma(1 + ^1/_2 iu - ^1/_2\nu)} \frac{\Gamma(^1/_2 + ^1/_2 iu)}{\Gamma(^1/_2 - ^1/_2 iu)} \tag{3.1}$$

Here it is necessary to use a well-known formula [11, p. 259 ] twice:

$$\int_0^\infty \frac{J_p(ax)\,dx}{x^{p-q}} = \frac{\Gamma(^1/_2 q + ^1/_2)}{2^{p-q}a^{q-p+1}\Gamma(p - ^1/_2 q + ^1/_2)} \tag{3.2}$$

In view of the analytic properties of the gamma function of Euler [12 ] we readily find that in the present case

$$\psi_+(u) = \frac{\Gamma(^1/_2 - ^1/_2 iu)}{\Gamma(^1/_2\nu - ^1/_2 iu)}, \qquad \psi_-(u) = \frac{2^{1-\nu}\Gamma(1 + ^1/_2 iu - ^1/_2\nu)}{\Gamma(^1/_2 + ^1/_2 iu)} \tag{3.3}$$

Using the well-known asymptotic representation of $\Gamma(z)$ at infinity [12 ] it may be verified that conditions (2.14) are satisfied.

Hence, the function $\chi_\zeta(z)$ is found. Substituting it into Formula (2.11) we obtain the solution of the contact-problem integral equation for the case of a general right-hand side. By means of a number of transformations this solution may be brought into the form of one of the known solutions [1,13 ].

At the end of the paper a simpler form will be found for the solution of the problem at hand with an arbitrary right-hand side. However, having only the solution $\chi_\zeta(x)$ for a special right-hand side, one can examine the majority of cases of practical interest in a contact problem.

We shall denote the right-hand side of Equation (2.7) corresponding to this solution by $f_\zeta(r)$. Thereby, on the basis of (2.8), we have the equation

$$f_\zeta(r) = a^{i\zeta}r^{-\nu-i\zeta} \tag{3.4}$$

We denote the contact stress in this case by $p_\zeta(r)$. It will be associated with the solution of Equation (2.9) by the relation

$$\chi_\zeta(x) = ae^{-x}p_\zeta(ae^{-x}), \qquad ae^{-x} = r \tag{3.5}$$

By the proposed method the solution of the contact problem permits one to find the magnitude of the force $P$ compressing the bodies without first determining the contact stresses and integrating them over the interval $(0,\ a)$.

Although this is true in the general case we shall restrict ourselves in the proof to the case (3.4) for which

$$P_\zeta = 2\pi \int_0^a r p_\zeta(r)\, dr \tag{3.6}$$

The generalized Fourier transformation of the function $\chi_\zeta(x)$ [8]

$$X_\zeta(u) = \int_0^\infty \chi_\zeta(x)\, e^{iux}\, dx$$

will have on the basis of (2.12) the form

$$X_\zeta(u) = i\,\frac{\psi_+(u)\,\psi_-(-\zeta)}{u+\zeta} \tag{3.7}$$

The compressive force $P_\zeta$ is very simply expressed in terms of this function

$$P_\zeta = 2\pi a X_\zeta(i) \tag{3.8}$$

In order to verify this it is necessary to consider the relationship

$$aX_\zeta(u+i) = a\int_0^\infty \chi_\zeta(x)\, e^{i(u+i)x}\, dx$$

substitute therein (3.5) and make an obvious change of the variable of integration; as a result one obtains

$$aX_\zeta(u+i) = \int_0^a r p_\zeta (a/r)^{iu}\, dr$$

Thus, taking (3.6) into account, (3.8) follows also. In view of (3.7) and (3.8) we find

$$P_\zeta = \frac{i\pi a}{i+\zeta}\,\frac{\Gamma(1-{}^1/_2 i\zeta + {}^1/_2\nu)}{\Gamma({}^1/_2 + {}^1/_2\nu)\,\Gamma({}^1/_2 - {}^1/_2 i\zeta)} \tag{3.9}$$

In order to obtain a formula for $p_\zeta(r)$ which will be convenient in the sequel we transform the right-hand side of (2.12). With this goal in mind and taking into account (3.3) we represent it in the form

$$\chi_\zeta(x) = \psi_-(-\zeta)\,[S_1(x) + S_2(x)] \tag{3.10}$$

where

$$S_1(x) = \frac{i}{2\pi}\int_\gamma \frac{\Gamma({}^1/_2 - {}^1/_2 iu)}{2i\Gamma(1+{}^1/_2\nu - {}^1/_2 iu)}\, e^{-iux}\, du \tag{3.11}$$

$$S_2(x) = \frac{i(\nu + i\zeta)}{2\pi}\int_\gamma \frac{\Gamma({}^1/_2 - {}^1/_2 iu)\, e^{-iux} du}{2(u+\zeta)\,\Gamma(1+{}^1/_2\nu - {}^1/_2 iu)} \tag{3.12}$$

Here we have used the identity

$$\frac{1}{u+\zeta}=\frac{1}{u+i\nu}+\frac{\nu+i\zeta}{(\nu-iu)(u+\zeta)}$$

and the well-known relation for the gamma function [12 ]

$$\Gamma(1+z)=z\Gamma(z) \tag{3.13}$$

The integral (3.11) is the sum of residues

$$S_1(x)=\sum_{n=0}^{\infty}\frac{(-1)^n e^{-(2n+1)x}}{n!\,\Gamma(^1/_2+{}^1/_2\nu-n)} \tag{3.14}$$

On the basis of the well-known relation

$$\Gamma(z)\Gamma(1-z)=\pi\operatorname{cosec} z \tag{3.15}$$

the following equality holds:

$$\pi\Gamma^{-1}(^1/_2+{}^1/_2\nu-n)=(-1)^n\sin(^1/_2-{}^1/_2\nu)\,\pi\Gamma(^1/_2-{}^1/_2\nu+n)$$

Taking into account the latter, as well as the series representation for the hypergeometric function [12 ], one may write

$$S_1(x)=\pi^{-1}\sin(^1/_2-{}^1/_2\nu)\,\pi\Gamma(^1/_2-{}^1/_2\nu)\,e^{-x}F(^1/_2-{}^1/_2\nu,\ \beta,\ \beta,\ e^{-2x})$$

whence [12, p. 329 ]

$$S_1(x)=\pi^{-1}\sin(^1/_2-{}^1/_2\nu)\,\pi\Gamma(^1/_2-{}^1/_2\nu)\,e^{-x}(1-e^{-2x})^{^1/_2\nu-{}^1/_2} \tag{3.16}$$

If one uses the indentity

$$(u+\zeta)^{-1}=-i\int_0^1\eta^{-iu-i\zeta-1}\,d\eta$$

then, after a change in the order of integration, Expression (3.12) takes on the form

$$S_2(x)=(\nu+i\zeta)\int_0^1\eta^{-i\zeta-1}\,d\eta\,\frac{1}{2\pi i}\int_\gamma\frac{\Gamma(^1/_2-{}^1/_2 iu)}{2\Gamma(1+{}^1/_2\nu-{}^1/_2 iu)}\left(\frac{e^{-x}}{\eta}\right)^{iu} i\,du$$

Making the change of variables $-iu = s$ in the inner integral and using the relation [14, p. 88 ]

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{\Gamma(\beta+{}^1/_2 s)}{\Gamma(\alpha+\beta+s/2)}\,x^{-s}ds=\begin{cases}2^{2\beta}\Gamma^{-1}(\alpha)(1-x^2)^{\alpha-1} & (0\leqslant x\leqslant 1)\\ 0 & (x>1)\end{cases}$$

we find

$$S_2(x) = e^{-x} \frac{\nu + i\zeta}{\Gamma(^1/_2 + {}^1/_2\nu)} \int_{e^{-x}}^{1} \eta^{-i\zeta-2} \left(1 - \frac{e^{-2x}}{\eta^2}\right)^{^1/_2\nu - ^1/_2} \tag{3.17}$$

By virtue of (3.5), (3.10), (3.16) and (3.17) we obtain the contact stress $p_\zeta(r)$

$$p_\zeta(r) = \frac{\psi_-(-\zeta)}{\Gamma(^1/_2 + {}^1/_2\nu)} \left[a^{-\nu}(a^2 - r^2)^{^1/_2\nu - ^1/_2} + (\nu + i\zeta)\, a^{i\zeta} \int_r^a \frac{(t^2 - r^2)^{^1/_2\nu - ^1/_2} \cdot}{t^{1+\nu+i\zeta}} dt\right] \tag{3.18}$$

**4.** In order to encompass most of the examples of axisymmetric contact problems for the case of a homogeneous body which are considered by Shtaerman in his monograph, it is sufficient to restrict ourselves to the case

$$z_1(r) + z_2(r) = Ar^\sigma$$

where $\sigma$ is an arbitrary positive number. In this case

$$f(r) = (c_1 + c_2)^{-1}(\alpha - Ar^\sigma)$$

and, therefore, on the basis of (3.4)

$$f(r) = \frac{a^\nu}{c_1 + c_2}[f_\zeta(r)]_{\zeta = i\nu} - \frac{Aa^{\nu+\sigma}}{c_1 + c_2}[f_\zeta(r)]_{\zeta = j\nu + i\sigma}$$

and hence

$$p(r),\ P = \frac{a^\nu}{c_1 + c_2}[p_\zeta(r),\ P_\zeta]_{\zeta = i\nu} - \frac{Aa^{\nu+\sigma}}{c_1 + c_2}[p_\zeta(r),\ P_\zeta]_{\zeta = i\nu + i\sigma} \tag{4.1}$$

From this, using the obtained formulas (3.18) and (3.19), we find

$$p(r) = \frac{2^{1-\nu}\Gamma^{-1}(^1/_2 + {}^1/_2\nu)}{c_1 + c_2}\left[\frac{\alpha}{\Gamma(^1/_2 + {}^1/_2\nu)} - A\frac{\Gamma(1 + {}^1/_2\sigma)\, a^\sigma}{\Gamma(^1/_2 + {}^1/_2\nu + {}^1/_2\sigma)}\right](a^2 - r^2)^{^1/_2\nu - ^1/_2} +$$

$$+ \frac{A}{c_1 + c_2}\frac{2^{1-\nu}\Gamma(1 + {}^1/_2\sigma)\,\sigma}{\Gamma(^1/_2 + {}^1/_2\nu)\,\Gamma(^1/_2 + {}^1/_2\nu + {}^1/_2\sigma)}\int_r^a (t^2 - r^2)^{^1/_2\nu - ^1/_2} t^{\sigma-1} dt \tag{4.2}$$

$$P = \frac{\pi 2^{2-\nu} a^{1+\nu}}{(c_1 + c_2)\,\Gamma(^1/_2 + {}^1/_2\nu)}\left[\frac{\alpha}{(1 + \nu)\,\Gamma(^1/_2 + {}^1/_2\nu)} - \frac{A\Gamma(1 + \sigma/2)\, a^\sigma}{(1 + \nu + \sigma)\,\Gamma(^1/_2 + {}^1/_2\nu + {}^1/_2\sigma)}\right] \tag{4.3}$$

From the condition that the contact stress is finite at $r = a$ we find

$$\alpha = A\frac{\Gamma(^1/_2 + {}^1/_2\nu)\,\Gamma(1 + {}^1/_2\sigma)}{\Gamma(^1/_2 + {}^1/_2\nu + {}^1/_2\sigma)}a^\sigma \tag{4.4}$$

Substituting (4.4) into (4.3) we find the radius of the area of contact

$$a^{1+\nu+\sigma} = \frac{P}{A}\frac{c_1+c_2}{2^{2-\nu}\pi}\frac{(1+\nu)(1+\nu+\sigma)}{\Gamma(1+{}^1/_2\sigma)}\Gamma({}^1/_2+{}^1/_2\nu)\Gamma({}^1/_2+{}^1/_2\nu+{}^1/_2\sigma) \quad (4.5)$$

By virtue of (4.5) and (4.2) the formula for the contact stress in this case may be transformed into the form

$$p(r) = \frac{P(1+\nu)(1+\nu+\sigma)}{2\pi a^2}\int\limits_{r/a}^{1}\left(t^2-\frac{r^2}{a^2}\right)^{{}^1/_2\nu-{}^1/_2} t^{\sigma-1}ds \quad (4.6)$$

We consider now separate particular cases. We arrive at the case of the indentation of a stamp with a plane base if we substitute into the above formulas $A = 0$, $\sigma = 0$, $c_2 = 0$, $c_1 = c$. Thereby, Formula (4.2) goes over into the formula obtained by Mossakovskii [13], and Formula (4.6) takes on the form

$$p(r) = \frac{P}{2\pi}\frac{\nu+1}{a^{\nu+1}}(a^2-r^2)^{{}^1/_2\nu-{}^1/_2}.$$

For $\nu = 0$ we obtain the Boussinesq formula.

Further, we examine the indentation of a cone into an elastic body which is close to a half-space. If the generators of the cone form the angle $\gamma$ with the axis of symmetry $z$, then [2, p. 43]

$$z_1(r) + z_2(r) = r\cot\gamma$$

Hence, setting $\sigma = 1$, $A = \cot\gamma$, in Formulas (4.4) to (4.6) we find

$$p(r) = \frac{P(1+\nu)(2+\nu)}{2\pi a^2}\int\limits_{r/a}^{1}\left(t^2-\frac{r^2}{a^2}\right)^{{}^1/_2\nu-{}^1/_2}dt$$

$$\alpha = \frac{\sqrt{\pi}}{2\tan\gamma}\frac{\Gamma({}^1/_2+{}^1/_2\nu)}{\Gamma(1+{}^1/_2\nu)}a, \qquad a^2 = \frac{P(c_1+c_2)(1+\nu)(2+\nu)}{2^{1-\nu}\cot\gamma\pi^{{}^3/_2}}\Gamma({}^1/_2+{}^1/_2\nu)\Gamma(1+{}^1/_2\nu)$$

We note that in contrast to the case of a homogeneous body ($\nu = 0$) the contact stress in this case does not have a singularity at the origin.

As in the case of homogeneous bodies for $z_1(r) + z_2(r) = Ar^{2n}$, i.e. for $\sigma = 2n$, the contact stress here can also be expressed in terms of elementary functions. Indeed, proceeding in the present case as in [2], we obtain

$$p(r) = \frac{(1+\nu)(1+\nu+2n)P}{2\pi a^2}\left[\frac{1}{\nu+2n-1}+\frac{2n-2}{(\nu+2n-1)(\nu+2n-3)}\left(\frac{r}{a}\right)^2+\cdots\right.$$
$$\left.\cdots+\frac{(2n-2)(2n-4)\ldots6\cdot4\cdot2}{(\nu+2n-1)(\nu+2n-3)\ldots(\nu+3)(\nu+1)}\left(\frac{r}{a}\right)^{2n-2}\right]\left(1-\frac{r^2}{a^2}\right)^{{}^1/_2+{}^1/_2\nu}$$

Formulas (4.4) and (4.5) correspondingly take on the forms

$$\alpha = A \cdot n! \frac{\Gamma(^1/_2 + ^1/_2\nu)}{\Gamma(^1/_2 + ^1/_2\nu + n)} a^{2n}$$

$$a^{2n+1} = \frac{P}{A} \frac{c_1 + c_2}{2^{2-\nu}\pi} \frac{(1+\nu)(1+\nu+2n)}{n!} \Gamma(^1/_2 + ^1/_2\nu)\,\Gamma(^1/_2 + ^1/_2\nu + n)$$

We shall give now another formula, different from (2.11), for the solution of the axisymmetrical contact-problem integral equation. In this we start with the following result of Krein. He has shown [15] that the solution of the integral equation of the type

$$\int_0^a k(r, s)\,\varphi(s)\,ds - \mu\varphi(r) = f(r) \qquad (0 \leqslant r \leqslant a) \tag{4.7}$$

is given by the formula

$$\begin{aligned}\varphi(r) = \frac{1}{M'(a)}\left[\frac{d}{da}\int_0^a q^*(s;\ a)\,f(s)\,ds\right] q(r;\ a) - \\ - \int_r^a q(r;\ u)\,\frac{d}{du}\left[\frac{1}{M'(u)}\,\frac{d}{du}\int_0^u q(s;\ u)\,f(s)\,ds\right] du\end{aligned} \tag{4.8}$$

where $q(r;\ a)$ is the solution of the integral equation (4.7) with $f(r) \equiv 1$, and $q^*(r;\ a)$ is the solution of the associated equation. In this case

$$M(a) = \int_0^a q(r;\ a)\,dr$$

We apply Formula (4.8) to the solution of the integral equation (2.4)*. With this goal in mind we represent it in the following form:

$$\int_0^a K(r, \rho)\,\rho p(\rho)\,d\rho = f(r) \tag{4.9}$$

where

$$K(r, \rho) = \rho^{-1-\nu} k_\nu(r/\rho) = \int_0^\infty J_0(s\rho)\,J_0(sr)\,s^\nu\,ds$$

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* The idea of applying Formula (4.8) to the elastic contact problem is due to Krein, who used it on the plane elastic contact problem in the case of a homogeneous half-plane. This material has not been published by him.

Taking as the unknown function

$$\varphi(r) = rp(r) \tag{4.10}$$

it is easy to verify that $K(r, \rho) = K(\rho, r)$ and therefore

$$q^*(r; a) = q(r; a).$$

The solution $q(r; z)$ of Equation (4.9) for $f(r) \equiv 1$ is not difficult to find by the use of the results obtained above. Indeed, in view of (2.8), (3.5), (3.18) and (4.10), it can be seen that

$$q(r; a) = a^\nu r\,[p_\zeta(r)]_{\zeta=i\nu} = 2^{1-\nu}\Gamma^{-2}(1/2 + 1/2\nu)\, r\,(a^2 - r^2)^{1/2\nu - 1/2} \tag{4.11}$$

whence

$$M(a) = \frac{2^{1-\nu}a^{\nu+1}}{(\nu+1)\Gamma^2(1/2 + 1/2\,\nu)}, \qquad M'(a) = \frac{2^{1-\nu}a^\nu}{\Gamma^2(1/2 + 1/2\,\nu)} \tag{4.12}$$

In addition it is not hard to verify that

$$\frac{d}{da}\left[\int_0^a \frac{sf(s)\,ds}{(a^2 - s^2)^{1/2 - 1/2\nu}}\right] = \frac{a^\nu}{\nu+1}\left[f(0) + a\int_0^1 \frac{f'(at)(1+\nu t^2)}{(1-t^2)^{1/2-1/2\nu}}\,dt\right] \tag{4.13}$$

Substituting (4.10), (4.11) and (4.12) into Formula (4.8) and taking into account (4.13), we find after some obvious transformations that

$$p(r) = \frac{2^{1-\nu}}{\Gamma^2\left(\frac{1+\nu}{2}\right)}\left[\frac{\gamma}{(a^2 - r^2)^{1/2 - 1/2\nu}} - \int_r^a \frac{u^{-\nu}\,du}{(a^2 - r^2)^{1/2 - 1/2\nu}}\int_0^u \frac{f'(s) + sf''(s)}{(u^2 - s^2)^{1/2 - 1/2\nu}}\left(1 + \nu\frac{s^2}{u^2}\right)ds\right] \tag{4.14}$$

where

$$\gamma = f(0) + a^{1-\nu}\int_0^a \frac{f'(s)}{(a^2 - s^2)^{1/2 - 1/2\nu}}\left(1 + \nu\frac{s^2}{a^2}\right)ds$$

In conclusion, we show that the method which has been presented here allows one to construct the exact solution of the axisymmetric problem when the form of surfaces of the contacting bodies is taken into account. Indeed, if one assumes contact along a circle in this case, the integral equation (1.6) will have the form

$$kp(r) + \int_0^a w(r, \rho)\,p(\rho)\,d\rho = w(r) \qquad (0 < r \leqslant a) \tag{4.15}$$

If one restricts oneself to the homogeneous half-space and assumes that a coefficient $k$, depending on the structure of the contacting surfaces, has the form $k = \kappa r$, $k$ = const, then, as a result of the change of variables (2.6) and because of (2.2) and (2.3), Equation (4.15) may be transformed into the following Wiener-Hopf integral equation of the second kind:

$$\chi(x) + \lambda \int_0^\infty k_0 (e^{-(x-\xi)}) \chi(\xi) \, d\xi = f(x), \qquad (0 \leqslant x < \infty) \tag{4.16}$$

where

$$\chi(x) = ae^{-x} p(ae^{-x}), \quad \lambda = 2(1 - \mu_0^2)(\varkappa E)^{-1}, \quad f(x) = \varkappa^{-1} w(ae^{-x})$$

As is well known, one can construct an exact solution of an integral equation of the type (4.16) [9], and hence one can obtain an exact solution of the problem which has been formulated.

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